

# New 16-PSK Trellis Codes for Fading Channels

Jun Du, Branka Vucetic and Lin Zhang

The School of Electrical Engineering  
 The University of Sydney, NSW 2006, Australia  
 Phone: 62-2-692-2154      Fax: 62-2-692-3847

## Abstract

Growth in satellite mobile communications leads to increasing requirements for high data rate transmission that can be met by more efficient modulation schemes ( $M > 8$ ). The 16-PSK trellis coded modulation technique is a very promising solution. In this paper, a class of new 16-PSK trellis codes with improved error rate are designed based on the criteria on fading channels.

## 1 Introduction

The expansion of satellite communications implies the use of higher radio frequencies (SHF and EHF regions where a major disadvantage associated with such systems is the signal attenuation due to fading. Most of the TCM codes constructed for Gaussian channels suffer significantly degradation on fading channels. It has been pointed out[2][3][4] that the codes for fading channels should be designed by using the criteria different from those for Gaussian channels.

Some efforts have been made to constructing new 8-PSK TCM codes for fading channels[3]. The growth in satellite communications, however, leads to increasing requirements for high data rate transmission that can be met by more efficient modulation schemes ( $M > 8$ ). So far, the efforts on construction of 16-PSK TCM codes were concentrated on Gaussian channels [1][5][6][7]. The problem of designing 16-PSK TCM codes for fading channels is studied in this paper. A class of new 16-PSK trellis codes with improved performance on fading channels are proposed based on the code design criteria for fading channels. The performances of the 16-PSK TCM systems on Rayleigh fading channels are evaluated both theoretically and by Monte Carlo simulations.

## 2 System Description

In the system under consideration, the digital source generates data digit stream at a rate of  $1/T_b$ . From the 3-input information sequence, the convolutional encoder generate 4-output coded sequence. The coded sequences are then interleaved and used to modulate 16-PSK waveform at a rate  $1/3T_b$  symbols/s according to symbol mapping rules. The channel is described by a slowly varying frequency non-selective Rayleigh fading. The receiver consists of a coherent demodulator followed by the deinterleaver and a Viterbi algorithm.

The channel is modelled by non-selective Rayleigh fading. We shall restrict attention to channels with slow amplitude variations relative to an elementary signaling interval and assume perfect phase tracking of the phase perturbation process. The interleaving depth is chosen to achieve ideal interleaving for a given fade rate in the simulations and assumed infinite in the analysis. In this model, the amplitude of the envelope of the received signal is multiplied by a random variable  $a$ , normalized so that  $E[a^2] = 1$  to ensure that the received signal energy is equal to the transmitted signal energy  $E_s$ . The probability distribution function of  $a$  is

$$p(a) = 2ae^{-a^2} \quad (1)$$

The mean of the random variable  $a$  is

$$m_a = \frac{\sqrt{\pi}}{2} \quad (2)$$

while the variance is given by

$$\sigma_a^2 = 1 - m_a^2 \quad (3)$$

### 3 Error Probability and Code Design Criteria

Figure 1 shows a 16-PSK signal constellation and the notation where  $\Delta_i$  denotes the Euclidean distance between channel symbols. Let  $C$  denote the set of all possible coded signal sequences  $\{z_N\}$ , where  $z_N$  is a coded symbol sequence of length  $N$ :

$$z_N = (z_1, z_2, \dots, z_i, \dots, z_N)$$

Let us suppose that  $v_N$ :

$$v_N = (v_1, v_2, \dots, v_i, \dots, v_N)$$

an element of  $C$ , is the transmitted sequence. Then  $v_i$  represents the MPSK symbol transmitted at time  $i$ . The corresponding received sequence at the input of the decoder is:

$$r_N = (r_1, r_2, \dots, r_i, \dots, r_N)$$

At time  $i$ , the received symbol  $r_i$  is related to the transmitted symbol  $v_i$  by:

$$r_i = a_i v_i + n_i$$

where  $a_i$  is a complex random variable equal to the envelope of the fading channel normalized to the transmitted signal and  $n_i$  is a sample of a zero mean complex additive white Gaussian noise (AWGN) with variance  $\sigma^2$  in each signal space coordinate. The corresponding received sequence  $r_N$  can be represented as

$$r_N = a_N \times v_N + n_N$$

where  $\times$  denotes vector multiplication.

In the Viterbi algorithm, the path metric  $m(r_N, z_N)$  is the sum of the branch metrics:

$$m(r_N, z_N) = \sum_{i=1}^N m(r_i, z_i)$$

The branch metric used in the algorithm is equal to the squared Euclidean distance:

$$m_i = |r_i - z_i|^2 \quad (4)$$

#### Definitions

In the context of fading communications, besides the minimum Euclidean distance,  $d_e$ , and the number of paths at distance  $d_e$  from the correct path,

$N(d_e)$ , the following parameters play important roles:

1) *Effective Code Length* (ECL): the minimum effective length taken over all error paths in  $C$  where the *Effective Length*, denoted by  $\delta$ , of an error path is the number of erroneous symbols on the error path, and can be expressed as

$$\delta = \sum_{i=0}^{M/2-1} w_i \quad (5)$$

where  $M$  is the number of signal points in the signal constellation and  $w_i$  are the weights associated with the corresponding nonzero symbol distances  $\Delta_i$  along the error path. The ECL of a TCM code is denoted by  $\delta_H$  in this paper.

2) *Minimum Product Distance*: the minimum product distance taken over all paths with the ECL where *Product Distance*, denoted by  $\Delta_P$ , is the product of the corresponding nonzero squared symbol distances along the error path and is given by

$$\Delta_P = \prod_{i=0}^{M/2-1} \Delta_i^{2w_i} \quad (6)$$

The minimum PD is denoted by  $d_p$  in this paper.

Similar to  $N(d_e)$ , we have  $N(d_p)$  associated with  $d_p$ , which is the number of the paths at product distance  $d_p$  from the correct path. In this paper both  $N(d_e)$  and  $N(d_p)$  are treated as average numbers in the same way as in [7] and [10] where the average is taken over all paths in the trellis.

#### Pairwise Error Probability

The pairwise error probability of choosing the path  $z_N$  instead of  $v_N$  (first error event probability) is equal to

$$P(v_N \rightarrow z_N) = \Pr\{m(r_N, z_N) < m(r_N, v_N)\}$$

or equivalently

$$P(v_N \rightarrow z_N) =$$

$$\Pr\{2\operatorname{Re}[(z_N - v_N)^* \cdot (r_N - v_N)] \geq m(z_N, v_N)\} \quad (7)$$

where  $*$  denotes the complex conjugate and  $\cdot$  denotes the scalar product of two vectors. The

squared Euclidean distance between the transmitted and error path is given by

$$d^2(z_N, v_N) = \sum_{i=1}^N |z_i - v_i|^2 = \sum_{i=1}^N d_i^2$$

Since  $d_i$  is equal to one of the  $\Delta_j$ ,  $0 \leq j < M/2$ , (Fig. 1)  $d^2(r_N, z_N)$  can be expressed as

$$d^2(r_N, z_N) = \sum_{i=0}^{M/2-1} w_i \Delta_i^2 \quad (8)$$

With infinite interleaving, the sequence  $a_N$  of fading amplitudes constitutes an independent and identically distributed sequence. Equation (7) can be rewritten as

$$\begin{aligned} P(v_N \rightarrow z_N) &= \Pr\{2\operatorname{Re}[(z_N - v_N)^* \cdot n_N] \geq \\ &d^2(z_N, v_N) - 2\operatorname{Re}[(z_N - v_N)^* \cdot (a_N \times v_N - v_N)]\} \end{aligned} \quad (9)$$

Let us suppose for simplicity that  $a_N$  is real and the modulation scheme is M-PSK. Then the right hand side of Eq.(9) can be simplified by noting that:

$$-2\operatorname{Re}[(z_N - v_N)^* \cdot v_N] = d^2(z_N, v_N)$$

and

$$-2\operatorname{Re}[(z_N - v_N)^* \cdot (a_N \times v_N)] = \sum_{i=1}^N a_i d_i^2$$

Then Eq.(9) can be rewritten as

$$\begin{aligned} P(v_N \rightarrow z_N) &= \\ \Pr\{2\operatorname{Re}[(z_N - v_N)^* \cdot n_N] &\geq \sum_{i=1}^N a_i d_i^2\} \end{aligned} \quad (10)$$

For a given realization of the fading variable  $a_N$ , the term on the left hand side of (10) is a zero-mean Gaussian random variable with variance  $4\sigma^2 d^2$  and the term on the right hand side is a constant which depends on  $a_N$ . The conditional pairwise error probability for a given realization of the fading random variable  $a_N$  is given by

$$P(v_N \rightarrow z_N | a_N) = Q\left(\frac{\sum_{i=1}^N a_i d_i^2}{2\sigma d}\right) \quad (11)$$

where  $Q(x)$  is the complementary error function defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt \quad (12)$$

In this analysis, we shall distinguish two cases, depending on the value of the ECL  $\delta_H$ . For large  $\delta_H$ , using the approximation  $Q(x) \leq 1/2\exp(-x^2/2)$  for  $x \gg 1$ ,  $P(v_N \rightarrow z_N)$  can be bounded by or equivalently

$$P(v_N \rightarrow z_N) = \\ P_{d,D} \leq \frac{1}{2} \frac{2\sigma d}{\sqrt{4\sigma^2 d^2 + \sigma_a^2 D^4}} \exp\left(-\frac{1}{2} \frac{m_a^2 d^4}{4\sigma^2 d^2 + \sigma_a^2 D}\right) \quad (13)$$

with

$$D = \sum_{i=1}^N d_i^4 \quad (14)$$

The formula (13) can be used for codes with large ECL for any SNR or for codes with medium ECL for small SNR where the influence of error paths with high effective lengths is significant. The bit error probability is upperbounded by

$$P_b \leq \frac{1}{k} \sum_{d,D} b_{d,D} P_{d,D} \quad (15)$$

where  $b_{d,D}$  is the total number of erroneous information bits on all paths at the Euclidean distance  $d$  from the transmitted path characterized by the distance  $D$  and  $k$  is the number of information bits in a symbol.

In the second case, when  $\delta_H$  is small, the Gaussian assumption is no longer valid and the probability has to be expressed as or equivalently

$$\begin{aligned} P(v_N \rightarrow z_N) &= \\ \int_{a_N} P(v_N \rightarrow z_N | a_N) p(a_1) \dots p(a_N) da_1 \dots da_N \end{aligned} \quad (16)$$

For large signal-to-noise ratios, we can write

$$P(v_N \rightarrow z_N) \leq \frac{1}{2} \prod_{i=1}^N \frac{8\sigma^2 d^2}{d_i^4} \quad (17)$$

or equivalently

$$P(v_N \rightarrow z_N) = P_{d,w} \leq \frac{1}{2} \frac{(8\sigma d)^{2\delta_H}}{\prod_{i=0}^{M/2-1} \Delta_i^{4w_i}} \quad (18)$$

It should be pointed out that the formula (18) is valid for short ECL codes and large SNR only.

The bit error probability is obtained by using the union bound technique as

$$P_b \leq \frac{1}{k} \sum_{d,w} b_{d,w} P_{d,w} \quad (19)$$

where  $b_{d,w}$  is the total number of erroneous information bits on all paths characterized by the Euclidean distance  $d$  and the set  $w$ .

## Code Design Criteria

The analytical results indicate that the design criteria for fading channels will depend on ECL values. For the codes with large ECL, the error probability (Eqs. (13)) is dominated by the minimum Euclidean distance, while for the codes with small ECL, Eq.(18) and Eq.(19) show that the error performance is mostly dependent on the ECL, the minimum PD and the number of erroneous bits on error paths,  $b_d$ .

To construct good codes for fading channel, it is very important to avoid having parallel transitions in the trellis. All the previously known 16-PSK and 16-QAM codes[6][5][7], however, have parallel transitions. An upper bound on the maximum ECL has been obtained[8]. Table 1 shows the upper bounds on ECL,  $L_m$ , for 16-ary trellis codes with constraint length  $\nu$ .

According to the possible values of ECL, to minimize error probability on fading channels one should attempt to

- when the possible ECL is small
  - (I-a) design the codes with maximum ECL and
  - (I-b) among the codes obtained form (I-a) maximize  $d_p$  first, and then maximize  $d_e$
  - (I-c) minimize  $N(d_p)$  and  $N(d_e)$  among the found codes
- when the possible ECL is large
  - (II-a) maximize  $d_e$
  - (II-b) minimize  $N(d_e)$  among the found codes

The criterion set I can be used to construct the codes with small possible ECL. It is possible to construct new 16-PSK TCM codes with no parallel transitions, though the minimum Euclidean distances are inferior to the known codes.

## 4 Code Construction

A computer search based on the defined criteria has been carried out in order to find out the best codes for fading channels. To minimize bit error rate, the Gray mapping (i.e., the neighboring symbols in a constellation differ only by one bit) is adopted in code construction.

A stack algorithm for calculating the minimum free Hamming distance of convolutional codes has been used for calculating the minimum free Euclidean distance of trellis codes by replacing the Hamming weight of channel symbols by the Euclidean weight[1]. This algorithm is referred to as Algorithm 1 in this paper.

A modified version of this algorithm, which enables to compute not only  $d_e$  but also  $N(d_e)$ , has been developed[9]. In the modified algorithm, additional information on the multiplicity of the merging paths to a state with the lowest Euclidean weight is added into the stack memory. The search does not terminate  $d_e$  is reached, but continues till all the paths at  $d_e$  are found. We refer to this modified algorithm as Algorithm 2 here.

In order to compute  $d_p$ , another modification of the previous algorithms has been made. The new version can compute the minimum PD among the paths of any specified effective length. We refer to the new algorithm which computes only  $d_p$  as Algorithm 3. The other new algorithm which computes  $d_p$  as well as  $N(d_p)$  is referred to as Algorithm 4.

From Table 1, one can see that the maximum possible ECLs for 16-ary trellis codes are not large. According to the code design criteria in Section 3, the computer search should be carried out following the criteria set I.

To make the search efficient, Algorithms 3, 1, 4 and 2 are used successively for searching the best codes in terms of maximum ECL,  $d_p$ ,  $d_e$ , minimum  $N(d_p)$  and  $N(d_e)$  among the codes obtained at each previous step.

Table 2 lists the parameters of some new 16-PSK codes, where Code  $F_i$  is  $2^i$ -state code proposed for fading channel. As references, the parameters of the corresponding Ungerboeck 16-PSK TCM codes given in [6] are also listed as Code  $U_i$ . All of the other known 16-PSK codes[5][7] have parallel transitions and therefore they behave similarly on fading channels. In the table, the parity-check polynomials are given in the octal form, and  $N$  and  $G$  denote the natural mapping and the Gary mapping respectively.

It can be seen from the table that the proposed codes for fading channels have the maximum ECL and maximum  $d_p$ , but smaller  $d_e$  than the corresponding  $U_i$  type codes. It should be noted that

the search for codes was only based on the first term of  $d_p$  and  $d_e$ . It is possible to achieve a slight improvement in error rate performance by optimizing the distance spectrum though it requires more complicated search algorithms.

## 5 Performance Evaluation

The performance evaluation in terms of bit error rate (BER) has been conducted. To compute the analytical bounds for the BER performance, a modified Viterbi algorithm has been implemented to compute distance spectrum. Concurrently, extensive Monte Carlo computer simulations of the schemes above were carried out.

Fig. 2 shows the analytical and simulation BER results for the 16-state code F4 compared with code U4, on Rayleigh channel. Fig. 3 illustrates the results for the 128-state code F7 compared with code U7. A summary of the performance of all proposed Fi type codes is presented in Fig. 4.

The performance results show that all the Ungerboeck codes have error floors above  $BER=10^{-4}$  on fading channel due to parallel transitions. The proposed codes achieve significant improvement. As the simulation results show, at the  $BER = 10^{-3}$  (which is an acceptable rate for digital voice transmission), the difference between the two classes of codes varies from 1.5 dB for the 256-state codes to 6 dB for the 128-state codes on Rayleigh fading channels. This difference is even larger at lower error rates.

## 6 Conclusions

The design criteria based on ECL and PD are applied to construction of new 16-PSK trellis codes with improved error rate performance on fading channels. The code performance is evaluated both analytically and by simulation. The comparison with the previously known codes of the same complexity shows that the proposed codes can achieve significant improvement in bit error rate. The new 16-PSK trellis codes will meet the requirements for high data rate transmission in future satellite communications.

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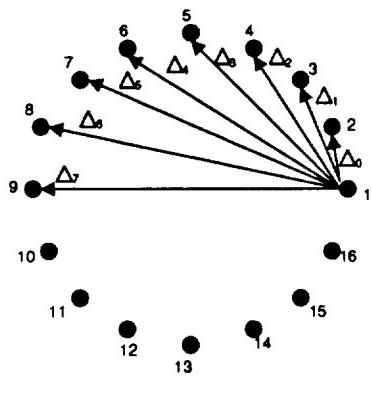


Figure 1: Notation for 16-PSK Modulation

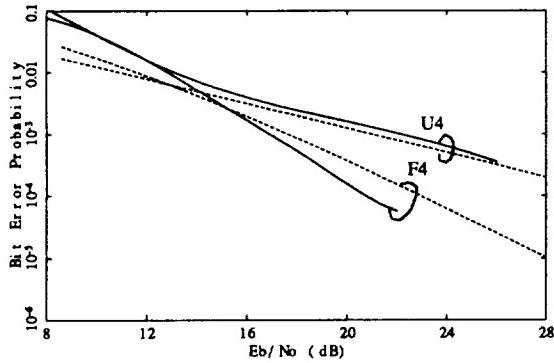


Figure 2: BER performance of the 16-state 16PSK TCM codes on Rayleigh channel  
Dash: Analysis; Solid: Monte Carlo simulation

<b>Table 1 Upper Bound on Maximum ECL .</b>
$\nu$ 2 3 4 5 6 7 8 9 10 11 ...

$L_m$	1	2	2	2	3	3	3	4	4	4	...
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**Table 2 16-PSK TCM codes .**

i.d.	$h^3$	$h^2$	$h^1$	$h^0$	$\delta_H$	$d_e$	$N(d_e)$	$d_p$	$N(d_p)$	Map.
U3	-	-	04	13	1	1.47	4.00	2.00	2.00	N
F3	15	17	05	13	2	0.89	0.19	0.61	0.25	G
U4	-	-	04	23	1	1.62	8.00	2.00	2.00	N
F4	37	21	15	33	2	0.89	0.34	2.47	0.50	G
U5	-	-	10	45	1	1.91	8.00	2.00	2.00	N
F5	75	73	47	57	2	1.48	0.77	13.7	1.00	G
U6	-	-	024	103	1	2.00	2.00	2.00	2.00	N
F6	165	175	115	143	3	1.48	1.90	0.68	1.00	G
U7	-	-	024	203	1	2.00	2.00	2.00	2.00	N
F7	337	343	307	211	3	1.48	0.03	4.69	2.00	G
U8	-	374	176	427	1	2.08	7.88	2.00	1.00	N
F8*	771	513	463	405	3	1.78	1.39	27.3	1.00	G

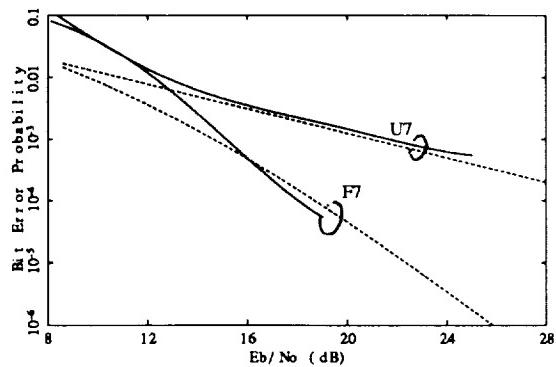


Figure 3: BER performance of the 128-state 16PSK TCM codes on Rayleigh channel  
Dash: Analysis; Solid: Monte Carlo simulation

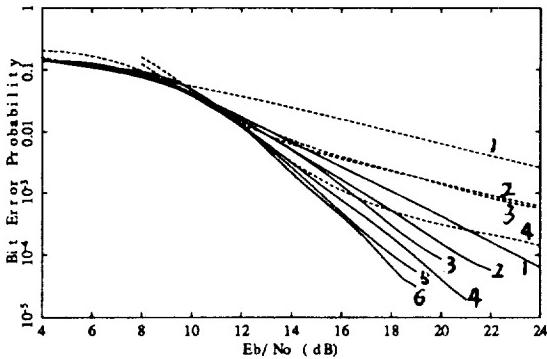


Figure 4: Summary of BER performance of 16PSK TCM codes on Rayleigh channel  
Dash 1: Uncoded 8-PSK; Dash 2 to 4: Codes U3,U7 and U8; Solid 1 to 6: Codes F3,F4,F5,F6,F7 and F8; All results are obtained by Monte Carlo simulations